If $2a_1$ be the major axis of the ellipse, $a_1^2=\frac{1}{4}d^2+b_1^2=\frac{1}{4}R^2$, or $2a_1=R$, and determining $4a_1^2+4b_1^2$.

392. Proposed by V. M. SPUNAR, M. and E. E., Chicago, Ill.

A tangent to a curve at any point P cuts the tangent and the normal at a fixed point O in the points M and N, and the rectangle $OMP'N$ is completed. Find the curve which is such that the triangle formed by the tangents at any three points P, Q, R is equal to the triangle formed by the corresponding points P', Q', R'.

No solution of this problem has been received.

CALCULUS.

316. Proposed by C. N. SCHMALL, New York City.

$$\int_0^\infty \frac{\cos ax}{1+x^2}dx=\tfrac{1}{2}\pi e^{-a}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx.$$

(From Bromwich, *Theory of Infinite Series*, p. 442, ex. 5, and also from Carslaw, *Fourier's Series*, p. 113, ex. 12.) Prove this by any method.

II. Solution by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Burghfield, England.

Lemma. $u=\underset{h\doteq 0}{L}[\sin h+\frac{1}{2}\sin 2h+\frac{1}{3}\sin 3h+\ldots]=\underset{h\doteq 0}{L}\frac{\pi-h}{2}$, the sum being taken between 2π and small values of h, $=\frac{1}{2}\pi$.

$\therefore u=\int_0^\infty \frac{\sin x}{x}dx=\frac{1}{2}\pi$, and it is also clear that $\int_0^\infty \frac{\sin ax}{x}dx=\frac{1}{2}\pi$.

Now let $U=\int_0^\infty \frac{x\sin ax\, dx}{1+x^2}$; $U-\frac{1}{2}\pi=\int_0^\infty \frac{x\sin ax\, dx}{1+x^2}-\int_0^\infty \frac{\sin ax}{x}$

$$=-\int_0^\infty \frac{\sin ax}{x(1+x^2)}dx\ldots(1).$$

Differentiating twice with respect to a,

$$\frac{d^2U}{da^2}=\int_0^\infty \frac{x\sin ax}{1+x^2}dx=U.$$

Multiplying by $\frac{dU}{d\,a}$, $\frac{d^2U}{da^2}\cdot\frac{dU}{d\,a}=\frac{U.dU}{da}$. $\frac{1}{2}\frac{d}{d\,a}\left(\frac{dU^2}{d\,a^2}\right)=\frac{1}{2}\frac{d\,(U^2)}{da}$, $\left(\frac{dU}{d\,a}\right)^2=U^2+\kappa$,

$$\frac{dU}{\sqrt{(U_2+\kappa)}}=da.$$

$\therefore \log[U+\sqrt{(U^2+\kappa)}]=a+\lambda,\ U+\sqrt{(U^2+\kappa)}=e^{a+\lambda}$.

Also, $\sqrt{(U^2+\kappa)}-U=\kappa\, e^{-(a+\lambda)}$, $2U=e^{a+\lambda}-\kappa\, e^{-(a+\lambda)}=Ce^{-a}+C'e^{a}$, where C, C', are constants. U not increasing indefinitely with a it follows that $C'=0$. When a is very small, (1) becomes

$$\underset{a\doteq 0}{L}\, U-\tfrac{1}{2}\pi=\underset{a\doteq 0}{L}\, -\int_0^\infty \frac{adx}{1+x^2}=\underset{a\doteq 0}{L}\, -\frac{a\pi}{2}=0;\ \therefore C=\tfrac{1}{2}\pi, \text{ and}$$

$$u=-\int_0^\infty \frac{x\sin axdx}{1+x^2}=\frac{\pi}{2}e^{-a}\ (a \text{ being positive}).$$

But $\displaystyle\int_0^\infty \frac{x\sin ax}{1+x^2}dx=\frac{\pi}{2}-u=\frac{\pi}{2}(1-e^{-a})$.

Differentiating with respect to a,

$$\int_0^\infty \frac{x\cos ax}{1+x^2}dx=\frac{\pi}{2}e^{-a}.$$

$\therefore$ etc. (Cf. Roberts' *Treatise on the Integral Calculus*, Part I, p. 181.)

317. Proposed by C. N. SCHMALL, New York City.

A generating line of a right circular cylinder passes through the center of a sphere. The diameter of the cylinder is less than the radius of the sphere. Show that the surface of the cylinder included within the sphere is given by an elliptic integral.

Solution by A. M. HARDING, Fayetteville, Arkansas.

Let a=diameter of cylinder; r=radius of sphere. Choose the generating line of cylinder for z-axis. Let equation of sphere and cylinder be

$$x^2+y^2+z^2=r^2 \text{ and } x^2+y^2=ax,$$

respectively. Then

$$\frac{A}{4}=\int\int\left[1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2\right]^{\frac{1}{2}}dzdx.$$

Eliminate y and obtain $z^2+ax=r^2$. Hence z-limits are 0 and $\sqrt{(r^2-ax)}$, x-limits are 0 and a.

From equation of cylinder, we find

$$\frac{\partial y}{\partial x}=\frac{a-2x}{2y},\ \frac{\partial y}{\partial z}=0.$$